OSCILLATION AND CONCENTRATION UNDER CONSTANT RANK CONSTRAINTS

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ABSTRACT. We provide some refinements of several results in (Fonseca, I. and Müller, S., 1999. $\mathcal{A}$-Quasiconvexity, Lower Semicontinuity, and Young Measures. SIAM journal on mathematical analysis). In particular, we refine a lower semi-continuity result and the decomposition lemma. Namely, we show that both the oscillation and concentration effects of weakly convergent sequences in L^p, $1 < p < \infty$, that satisfy a PDE constraint of constant rank have gradient structure. We also characterize the generalized Young measures arising from such sequences by duality with $\mathcal{A}$-quasiconvex functions.

1. INTRODUCTION

Young measures are very useful functional analytic and measure theoretic tools to describe the effective limits of energy functionals

$$v \in \mathrm{L}^p(\Omega, \mathbb{V}) \mapsto \int_\Omega F(x, v(x))\mathrm{d}x, \quad |F(x,z)| \leqslant c(1+|z|)^p. \tag{1.1}$$

In their original formulation in [40] (see also [8, 4, 32, 6, 10]), they were used to describe oscillation phenomena in the Calculus of Variations. They have been subsequently used in the study of partial differential equations modeling numerous problems in continuum mechanics [38, 7, 39, 13]. Subsequent extensions, such as the DiPerna–Majda measures [14], were developed to also account for concentration effects. Oscillation and concentration are the only obstructions to strong convergence in Lebesgue spaces, so their interaction the with (lower semi-)continuity properties of the non-linear functionals (1.1) is fundamental.

In particular, in accordance with the framework of compensated compactness [38, 39, 30] we will couple the functional (1.1) with a linear differential constraint. This framework has been studied extensively in the past century, as can be seen from the works [29, 5, 11, 17, 24, 21, 12, 3, 19, 25], to name a few.

The main objective of the present work is to give a precise and robust characterization of L^p-Young measures, $1 < p < \infty$, generated by sequences satisfying a constant rank constraint by duality with Jensen inequalities for $\mathcal{A}$-quasiconvex functions, a terminology that we now begin to explain. Firstly, our notion of Young measures is closely related to that of Alibert–Bouchitté [1], see also [27, 37, 35]. The result is analogous to that of [18], where varifolds are used, in the case of weak convergence of gradients. A related result to ours, pertaining to DiPerna–Majda measures, was proved in [15]. Such characterizations for (oscillation) Young measures were first proved in [21, 22, 17]. Secondly, the correlation between lower semi-continuity and $\mathcal{A}$-quasiconvexity is a recurring theme, see [29, 11, 17, 3] and the related results on weak continuity [39, 30, 19].

Our analysis is based on two main parts. On one hand, we prove a refined Decomposition Lemma 1.1, which separates the gradient structure of oscillation and concentration effects. This extends similar results in [24, 22, 18, 17, 15, 34] and substantiates the fact asserted in the introduction of [34] (see also [19]) that the study of functionals defined on constant rank constrained sequences can be reduced to the study of functionals depending on (higher order) gradients, see [28]. On the other hand, we show that the method introduced recently in [25] is robust enough to enable us to construct generating sequences with ease.

We now begin a more technical description of our results. Throughout this paper, $\Omega \subset \mathbb{R}^n$ denotes an open and bounded set with negligible boundary, $\mathscr{L}^n(\partial\Omega) = 0$. The linear differential constraint with which we couple the functional (1.1) will be given by a (vectorial) linear partial differential operator

$$\mathcal{A}v := \sum_{|\alpha|=\ell} A_\alpha \partial^\alpha v \quad \text{for } v\colon \mathbb{R}^n \to \mathbb{V}, \quad \text{where } A_\alpha \in \mathrm{Lin}(\mathbb{V},\mathbb{W}) \text{ whenever } |\alpha| = \ell. \tag{1.2}$$

Here $\mathbb{V}$, $\mathbb{W}$ are finite dimensional inner product spaces over $\mathbb{R}$.

We will always assume that $\mathcal{A}$ satisfies the *constant rank condition* [36, 30, 17], that there exists $r \in \mathbb{N}_0$ such that

$$\operatorname{rank} \mathcal{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{where } \mathcal{A}(\xi) := \sum_{|\alpha|=\ell} \xi^\alpha A_\alpha, \tag{1.3}$$

as well as the *spanning wave cone condition*

$$\operatorname{span} \Lambda_{\mathcal{A}} = \mathbb{V}, \quad \text{where } \Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{S}^{n-1}} \ker \mathcal{A}(\xi). \tag{1.4}$$

Both of these conditions are standard for the type of problem we are looking at.

It was shown in [34] that there exists another constant rank operator $\mathcal{B}$ such that

$$\ker \mathcal{A}(\xi) = \operatorname{im} \mathcal{B}(\xi) \quad \text{for } \xi \neq 0 \quad \text{where } \mathcal{B}u = \mathbf{T}(D^k u) \quad \text{for } u\colon \mathbb{R}^n \to \mathbb{U}, \tag{1.5}$$

where k is the order of $\mathcal{B}$ and $\mathbf{T} \in \mathrm{Lin}(\mathrm{SLin}^k(\mathbb{R}^n, \mathbb{U}), \mathbb{V})$ is the tensor of coefficients of $\mathcal{B}$. We call such $\mathcal{B}$ a *potential operator* for $\mathcal{A}$. We refer the reader to Section 2.1 for the definition of p-Young measures.

Our first result is the following enhanced decomposition lemma:

Lemma 1.1. *Let $\mathcal{A}$ as in (1.2) be a constant rank operator with potential operator $\mathcal{B}$ such that (1.5) holds. Let $1 < p < \infty$ and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \to \mathcal{A}v \text{ in } \mathrm{W}^{-\ell,p}(\Omega, \mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu} = \big((\nu_x)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar{\Omega}}\big)$. Then there exist sequences (u_j), $(\tilde{u}_j) \subset \mathrm{C}_c^\infty(\Omega, \mathbb{U})$ and $(\tilde{b}_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ such that

$$\begin{aligned}
&v_j = v + \mathcal{B}u_j + \mathcal{B}\tilde{u}_j + \tilde{b}_j,\\
&\mathcal{B}u_j,\, \mathcal{B}\tilde{u}_j,\, \tilde{b}_j \rightharpoonup 0 \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}),\\
&(D^k u_j) \text{ is } p\text{-uniformly integrable},\\
&D^k \tilde{u}_j \to 0 \text{ in } \mathscr{L}^n\text{-measure},\\
&\tilde{b}_j \to 0 \text{ in } \mathrm{L}^p_{\mathrm{loc}}(\Omega, \mathbb{V}),
\end{aligned}$$

and, moreover, in $\mathrm{Y}^p(\Omega, \mathbb{V})$,

$$\begin{aligned}
&(v + \mathcal{B}u_j) \text{ generates } \left((\nu_x)_{x\in\Omega},\, 0,\, \texttt{n/a}\right),\\
&(\mathcal{B}\tilde{u}_j) \text{ generates } \left((\delta_0)_{x\in\Omega},\, \lambda \llcorner \Omega,\, (\nu_x^\infty)_{x\in\Omega}\right),\\
&(\tilde{b}_j) \text{ generates } \left((\delta_0)_{x\in\Omega},\, \lambda \llcorner \partial\Omega,\, (\nu_x^\infty)_{x\in\partial\Omega}\right).
\end{aligned}$$

In fact, a decomposition of the form $v_j = \mathcal{B}u_j + b_j$ where $(\mathcal{B}u_j)$ captures the oscillation and (b_j) captures the concentration in $\bar{\Omega}$ is also possible under a weaker constraint, see Lemma 3.1. In that case, we can improve one of the main results in [17] concerning lower semi-continuity for energy functionals arising from integrands of p-qrowth, i.e, Borel measurable maps $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ that satisfy

$$|F(x,z)| \leqslant c(1 + |z|)^p \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega \text{ and all } z \in \mathbb{V}. \tag{1.6}$$

Our result is:

Theorem 1.2. *Suppose that $\mathcal{A}$ satisfies conditions* (1.3) *and* (1.4). *Let $1<p,q<\infty$ and $F\colon \Omega\times\mathbb{V}\to[0,\infty)$ be a normal integrand of p-growth, i.e., satisfying* (1.6). *Suppose that $z\mapsto F(x,z)$ is $\mathcal{A}$-quasiconvex for $\mathscr{L}^n$-a.e. $x\in\Omega$. Then*

$$\left.\begin{aligned} v_j &\rightharpoonup v && \text{in } \mathrm{L}^p(\Omega,\mathbb{V})\\ \mathcal{A}v_j &\to \mathcal{A}v && \text{in } \mathrm{W}^{-\ell,q}(\Omega,\mathbb{W})\end{aligned}\right\}\implies \liminf_{j\to\infty}\int_\Omega F(x,v_j(x))\mathrm{d}x\geqslant\int_\Omega F(x,v(x))\mathrm{d}x. \tag{1.7}$$

By normal integrand we mean a jointly Borel measurable function that is lower semicontinuous in the second variable. We recall that an integrand as above is said to be $\mathcal{A}$-quasiconvex if, for $\mathscr{L}^n$-a.e. $x_0\in\Omega$, we have at all $z\in\mathbb{V}$ that

$$F(x_0,z)\leqslant \fint_{\mathbb{T}^n} F(x_0,z+v(x))\mathrm{d}x,\quad \text{for all } v\in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}) \text{ with } \mathcal{A}v=0,\ \fint_{\mathbb{T}^n} v(x)\mathrm{d}x=0.$$

For an autonomous integrand $f\colon\mathbb{V}\to\mathbb{R}$ satisfying (1.6), we define the *upper recession function* by

$$f_p^\infty(z):=\limsup_{(z',t)\to(z,\infty)}\frac{f(tz')}{t^p}\quad \text{for } (x,z)\in\Omega\times\mathbb{V}. \tag{1.8}$$

We have the following characterization result:

Theorem 1.3. *Suppose that $\mathcal{A}$ satisfies conditions* (1.3) *and* (1.4). *Let $\boldsymbol{\nu}\in \mathrm{Y}^p(\Omega,\mathbb{V})$.*

If $\boldsymbol{\nu}=:\big((\nu_x)_{x\in\Omega},\,\lambda,\,(\nu_x^\infty)_{x\in\bar\Omega}\big)$ is generated by a sequence $(v_j)\subset\mathrm{L}^p(\Omega,\mathbb{V})$ such that $(\mathcal{A}v_j)$ is strongly compact in $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{V})$, then

$$\left.\begin{aligned}&\langle f,\nu_x\rangle\geqslant f(\bar\nu_x) && \text{for } \mathscr{L}^n\text{-a.e. } x\in\Omega,\\ &\langle f_p^\infty,\nu_x^\infty\rangle\geqslant 0 && \text{for } \lambda\text{-a.e. } x\in\Omega,\end{aligned}\right.\quad \text{for all } \mathcal{A}\text{-quasiconvex } f \text{ satisfying (1.6).} \tag{1.9}$$

Conversely, suppose that $\lambda(\partial\Omega)=0$ and write $v(x):=\bar\nu_x$. Let $\mathcal{B}$ be a potential operator for $\mathcal{A}$, i.e. suppose that (1.5) *holds. Suppose also that the inequalities* (1.9) *hold. Then there exist sequences (u_j), $(\tilde u_j)\subset\mathrm{C}_c^\infty(\Omega,\mathbb{V})$ such that:*

$$\begin{aligned}&(v+\mathcal{B}u_j+\mathcal{B}\tilde u_j) \text{ generates } \boldsymbol{\nu},\\ &(D^k u_j) \text{ is p-uniformly integrable,}\\ &D^k\tilde u_j\to 0 \text{ in measure.}\end{aligned}$$

This shows, once again, that $\mathcal{A}$-quasiconvexity is intrinsic to weak convergence of PDE constrained sequences. In fact, we can say more: with $\mathbf{T}$ as in (1.5) and under (1.4), it was shown in [26] that f is $\mathcal{A}$-quasiconvex if and only if $f\circ\mathbf{T}$ is k-quasiconvex. Coupled with the properties of $(D^k u_j)$, $(D^k\tilde u_j)$ in the result above, we can say that weak-L^p convergence of constant rank constrained sequences reduces to convergence of higher order gradients.

2. Preliminaries

As general notation, we will use $Q_r(x)$ for the open cube of radius $r/2>0$ centered at $x\in\mathbb{R}^n$. $B_\mathbb{V}$ will denote the open unit ball in a normed linear space $\mathbb{V}$, whereas S_V will denote the unit sphere. We will write $\mathcal{M}(E)$ for the space of Radon measures defined on a locally compact Hausdorff space E, equipped with the Borel σ-algebra, $\mathcal{M}^+(E)$ for the cone of positive such measures, and $\mathcal{M}_1^+(E)$ for the space of probability measures.

Our convention for the Fourier transform is

$$\mathscr{F}v\equiv\hat v\colon \xi\in\mathbb{R}^n\mapsto\int_{\mathbb{R}^n} v(x)\mathrm{e}^{-\mathrm{i}\,x\cdot\xi}\mathrm{d}x$$

defined for Schwartz functions v and extended by duality to tempered distributions. On the torus $\mathbb{T}^n$ we use the obvious analogue for the Fourier coefficients.

Throughout the text, $\Omega\subset\mathbb{R}^n$ denotes a bounded and open set with $\mathscr{L}^n(\partial\Omega)=0$. Unless otherwise specified, $1<p<\infty$ will denote a Lebesgue exponent.

2.1. **Generalized Young measures.** The presentation here loosely follows the recent lecture notes [26]. Let $1 < p < \infty$. Consider the space of integrands

$$\mathbb{E}_p(\Omega, \mathbb{V}) := \left\{ \Phi \in \mathrm{C}(\Omega \times \mathbb{V}) \colon \Phi_p^\infty(x, z) := \lim_{t\to\infty,\, x'\to x} \frac{\Phi(x', tz)}{t^p} \in \mathbb{R} \text{ uniformly in } \bar\Omega \times S_{\mathbb{V}} \right\},$$

which is naturally equipped with the norm

$$\|\Phi\|_{\mathbb{E}_p} := \sup_{(x,z)\in\Omega\times\mathbb{V}} \frac{|\Phi(x,z)|}{(1+|z|)^p}.$$

When no ambiguity can arise, we may write $\mathbb{E}_p := \mathbb{E}_p(\Omega, \mathbb{V})$. It will thus be convenient to work with the coordinate transformations

$$S \colon \hat z \in B_{\mathbb{V}} \mapsto \frac{\hat z}{1 - |\hat z|} \in \mathbb{V}, \quad S^{-1} \colon z \in \mathbb{V} \mapsto \frac{z}{1+|z|} \in B_{\mathbb{V}},$$

where $B_{\mathbb{V}}$ denotes the open unit ball in $\mathbb{V}$. With this notation, the space of integrands $\mathbb{E}_p(\Omega, \mathbb{V})$ can be identified with $\mathrm{C}(\overline{\Omega \times B_{\mathbb{V}}})$ via the linear isometric isomorphism

$$(T_p\Phi)(x, \hat z) := (1 - |\hat z|)^p \Phi\left(x, \frac{\hat z}{1-|\hat z|}\right), \quad \text{for } x \in \Omega,\ \hat z \in B_{\mathbb{V}}.$$

It follows that its adjoint, $T_p^* \colon \mathbb{E}_p(\Omega, \mathbb{V})^* \to \mathrm{C}(\overline{\Omega \times B_{\mathbb{V}}})^* \cong \mathcal{M}(\overline{\Omega \times B_{\mathbb{V}}})$ is also a linear isometric isomorphism. We embed $\mathrm{L}^p(\Omega, \mathbb{V})$ into $\mathbb{E}_p^*$ via

$$\boldsymbol{\varepsilon}_v(\Phi) := \int_\Omega \Phi(x, v(x))\mathrm{d}x \leqslant \|\Phi\|_{\mathbb{E}_p} \int_\Omega (1+|v|)^p \mathrm{d}x \leqslant 2^{p-1}\|\Phi\|_{\mathbb{E}_p}(|\Omega| + \|v\|_{\mathrm{L}^p}^p),$$

so that, by the sequential Banach–Alaoglu theorem, we can conclude that bounded L^p sequences are weakly-* compact in $\mathbb{E}_p^*$ under the above identification. In particular, if (v_j) is bounded in $\mathrm{L}^p(\Omega)$, we know that along a subsequence we have $\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \boldsymbol{\nu}$ in $\mathbb{E}_p(\Omega, \mathbb{V})^*$. We define $\mu := (T_p^{-1})^*\boldsymbol{\nu} \in \mathcal{M}(\overline{\Omega \times B_{\mathbb{V}}})$ and write for $\Phi \in \mathbb{E}_p$

$$\begin{aligned} \langle\!\langle \Phi, \boldsymbol{\nu} \rangle\!\rangle &:= \langle \Phi, \boldsymbol{\nu} \rangle_{\mathbb{E}_p, \mathbb{E}_p^*} = \langle T_p\Phi, \mu \rangle \\ &= \int_{\bar\Omega\times B_{\mathbb{V}}} (1-|\hat z|)^p \Phi\left(x, \frac{\hat z}{1-|\hat z|}\right) \mathrm{d}\mu(x.\hat z) + \int_{\bar\Omega\times S_{\mathbb{V}}} \Phi_p^\infty(x, \hat z)\mathrm{d}\mu(x, \hat z). \end{aligned}$$

From this formula we derive two necessary conditions for the weakly-* limits of $\boldsymbol{\varepsilon}_{v_j}$, namely that $\mu \geqslant 0$ in the sense of $\mathcal{M}(\overline{\Omega \times B_{\mathbb{V}}})$ and

$$\int_\Omega \varphi(x)\mathrm{d}x = \int_{\bar\Omega\times B_{\mathbb{V}}} \varphi(x)(1-|\hat z|)^p \mathrm{d}\mu(x, \hat z) \text{ for all } \varphi \in \mathrm{C}(\bar\Omega). \tag{2.1}$$

Conversely, these conditions are sufficient to enable us to disintegrate μ into appropriately parametrized (generalized Young) measures that detect both oscillation and concentration behavior of an L^p-weakly convergent sequence (v_j). We define:

Definition 2.1. *A parametrized measure* $\boldsymbol{\nu} = \big((\nu_x)_{x\in\Omega},\ \lambda,\ (\nu_x^\infty)_{x\in\bar\Omega}\big)$ *is said to be an* L^p-Young measure *(or* p-Young measure*) whenever*

(a) $(\nu_x)_{x\in\Omega} \subset \mathcal{M}_1^+(\mathbb{V})$ *is weakly-* $\mathscr{L}^n$-measurable (the* oscillation measure*).*
(b) $\lambda \in \mathcal{M}^+(\bar\Omega)$ *(the* concentration measure*).*
(c) $(\nu_x^\infty)_{x\in\bar\Omega} \subset \mathcal{M}_1^+(\mathbb{V})$ *is weakly-* λ-measurable (the* concentration-angle measure*).*
(d) $\int_\Omega \int_{\mathbb{V}} |z|^p \mathrm{d}\nu_x(z)\mathrm{d}x < \infty$ *(the* moment condition *holds).*

Then $\boldsymbol{\nu}$ *acts linearly on* $\mathbb{E}_p(\Omega, \mathbb{V})$ *via*

$$\langle\!\langle \Phi, \boldsymbol{\nu} \rangle\!\rangle := \int_\Omega \int_{\mathbb{V}} \Phi(x, \cdot)\mathrm{d}\nu_x \mathrm{d}x + \int_{\bar\Omega} \int_{S_{\mathbb{V}}} \Phi_p^\infty(x, \cdot)\mathrm{d}\nu_x^\infty \mathrm{d}\lambda(x) \quad \textit{for } \Phi \in \mathbb{E}_p(\Omega, \mathbb{V}).$$

We write $\mathrm{Y}^p(\Omega, \mathbb{V})$ *(or simply* Y^p*) for the set of all such* $\boldsymbol{\nu}$.

It is then easy to check that a Young measure ν actually lies in $\mathbb{E}_p^*$ and, moreover, that the inclusion $\mathrm{Y}^p \subset \mathbb{E}_p^*$ is strict. We have the disintegration theorem:

Theorem 2.2. $\mathrm{Y}^p(\Omega, \mathbb{V}) = T_p^*\{\mu \in \mathcal{M}^+(\overline{\Omega \times B_{\mathbb{V}}})\colon$ *equation* (2.1) *holds*$\}$.

The description of Y^p can be pushed further: as a consequence of Theorem 1.3 with $\mathcal{A} \equiv 0$ and $\mathcal{B} = \mathrm{Id}$, we can prove that any $\boldsymbol{\nu} \in \mathrm{Y}^p$ can be obtained as a weakly-* limit in $\mathbb{E}_p^*$ of *elementary Young measures* $\boldsymbol{\varepsilon}_v = \big((\delta_{v(x)})_{x\in\Omega},\, 0,\, \mathtt{n/a}\big) \in \mathrm{Y}^p$. This fact can be proved directly, but will not be used in the sequel.

Coming back to Theorem 2.2, it implies that Y^p is weakly-* closed in $\mathbb{E}_p^*$ and convex. Collecting, we obtain the fundamental weak compactness result that we will use:

Theorem 2.3 (FTYM)**.** *Let* $(v_j)_j$ *be a bounded sequence in* $\mathrm{L}^p(\Omega, \mathbb{V})$*. Then there exists* $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ *such that, along a subsequence,* $\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \boldsymbol{\nu}$ *in* $\mathbb{E}_p(\Omega, \mathbb{V})^*$*, i.e.,*

$$\lim_{j\to\infty} \int_\Omega \Phi(x, v_j(x))\mathrm{d}x = \int_\Omega \int_{\mathbb{V}} \Phi(x, z)\mathrm{d}\nu_x(z)\mathrm{d}x + \int_{\bar{\Omega}} \int_{S_{\mathbb{V}}} \Phi_p^\infty(x, z)\mathrm{d}\nu_x^\infty(z)\mathrm{d}\lambda(x)$$

for all $\Phi \in \mathbb{E}_p(\Omega, \mathbb{V})$*.*

By taking $\Phi(x, z) = \varphi(x) z_i$ for $\varphi \in \mathrm{C}(\bar{\Omega})$, we see that $v(x) := \bar{\nu}_x := \langle \mathrm{id}, \nu_x\rangle$ for $\mathscr{L}^n$-a.e. $x \in \Omega$, since $\Phi_p^\infty \equiv 0$. It follows that $v_j \rightharpoonup v$ in $\mathrm{L}^p(\Omega, \mathbb{V})$. We will refer to the map v as the *barycentre* of $\boldsymbol{\nu}$.

One can test for weakly-* convergence in $\mathbb{E}_p^*$ with fewer integrands:

Lemma 2.4. *There exists a countable family* $\{\varphi\otimes f\colon \varphi \in \mathrm{Lip}(\Omega),\ f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{V})\cap\mathbb{E}_p(\Omega, \mathbb{V})\}$ *whose span is dense in* $\mathbb{E}_p(\Omega, \mathbb{V})$*. Moreover,*

$$|f(z_1) - f(z_2)| \leqslant c\|T_p f\|_{\mathrm{Lip}(B_{\mathbb{V}})}|z_1 - z_2|(1 + |z_1| + |z_2|)^{p-1} \quad \text{for } z_1,\, z_2 \in \mathbb{V},$$

where $\|g\|_{\mathrm{Lip}} := \|g\|_{\mathrm{L}^\infty} + \|Dg\|_{\mathrm{L}^\infty}$*.*

The main use of p-Young measures is that they efficiently separate the oscillation and L^p-concentration effects:

Theorem 2.5. *Let* $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ *generate* $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$*. Then* $v_j \to v$ *in* $\mathscr{L}^n$*-measure if and only if* $v(x) = \bar{\nu}_x$ *and* $\nu_x = \delta_{\bar{\nu}_x}$ *for* $\mathscr{L}^n$*-a.e.* $x \in \Omega$*.*

Moreover, let $\tilde{v}_j\colon \Omega \to \mathbb{V}$ *be measurable, such that* $v_j - \tilde{v}_j \to 0$ *in* $\mathscr{L}^n$*-measure. Then*

$$\int_\Omega \varphi(x) F(\tilde{v}_j(x))\mathrm{d}x \to \int_\Omega \varphi(x)\langle \nu_x, F\rangle \mathrm{d}x \quad \text{for } \varphi \in \mathrm{C}(\bar{\Omega}),\ F \in \mathrm{C}_c(\mathbb{V}).$$

Theorem 2.6. *Let* $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ *generate* $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ *and* $\Phi \in \mathbb{E}_p(\Omega, \mathbb{V})$*. Then* $(\Phi(\,\bullet\,, v_j))$ *is uniformly integrable if and only if*

$$\langle |\Phi_p^\infty(x, \bullet)|, \nu_x^\infty\rangle = 0 \qquad \text{for } \lambda\text{-a.e. } x \in \bar{\Omega}.$$

In particular, (v_j) *is p-uniformly integrable if and only if* $\lambda \equiv 0$*.*

Furthermore, if $(\tilde{v}_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ *generate* $\tilde{\boldsymbol{\nu}} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ *is such that* $(v_j - \tilde{v}_j)$ *is p-uniformly integrable, then* $\lambda_{\boldsymbol{\nu}} = \lambda_{\tilde{\boldsymbol{\nu}}} (=: \lambda)$ *and* $\nu_x^\infty = \tilde{\nu}_x^\infty$ *for* λ*-a.e.* $x \in \bar{\Omega}$*.*

Finally, we have the following classical lower semi-continuity result concerning (rough) integrands that are bounded from below and their interaction with the oscillation measure:

Proposition 2.7. *Let* $1 < p < \infty$ *and* $F\colon \Omega \times \mathbb{V} \to [0, \infty)$ *be a normal integrand of p-growth* (1.6)*. Let* $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ *generate* $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$*. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x \geqslant \int_\Omega \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x.$$

2.2. **Linear differential operators.** We will work with linear homogeneous partial differential operators with constant coefficients

$$\mathcal{A}v := \sum_{|\alpha|=\ell} A_\alpha \partial^\alpha v, \quad \mathcal{B}u := \sum_{|\beta|=k} B_\beta \partial^\beta u,$$

defined respectively for $\mathbb{V}$, $\mathbb{U}$-valued functions on $\mathbb{R}^n$ and having coefficients $A_\alpha \in \mathrm{Lin}(\mathbb{V},\mathbb{W})$, $B_\beta \in \mathrm{Lin}(\mathbb{U},\mathbb{V})$. These have characteristic polynomials

$$\mathcal{A}(\xi) := \sum_{|\alpha|=\ell} \xi^\alpha A_\alpha \in \mathrm{Lin}(\mathbb{V},\mathbb{W}), \quad \mathcal{B}(\xi) := \sum_{|\beta|=k} \xi^\beta B_\beta \in \mathrm{Lin}(\mathbb{U},\mathbb{V})$$

for $\xi \in \mathbb{R}^n$. We say that $\mathcal{A}$ satisfies the constant rank condition if there exists an integer r such that

$$\operatorname{rank} \mathcal{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

It was recently showed in [34] that for each such $\mathcal{A}$ there exists $\mathcal{B}$ as above such that the exact relation

$$\ker \mathcal{A}(\xi) = \ker \mathcal{B}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

This result was established by showing that the $\xi \mapsto \mathcal{A}^\dagger(\xi)$ defines a $(-\ell)$-homogeneous rational function that is smooth away from zero. Here $M^\dagger$ denotes the Moore–Penrose generalized inverse, see [9] for details.

As a consequence of the existence of $\mathcal{B}$, it was shown in [34] that we have the implication

$$\mathcal{A}v = 0 \quad \text{for } v \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}),\ \hat{v}(0)=0 \implies v = \mathcal{B}u \quad \text{for some } u \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{U}).$$

Another consequence is the identity

$$\mathrm{Id}_\mathbb{V} = \mathcal{B}(\xi)\mathcal{B}^\dagger(\xi) + \mathcal{A}^*(\xi)(\mathcal{A}^*)^\dagger(\xi), \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\},$$

which was used in [19, 20] to obtain Helmholtz-type decompositions such as:

Proposition 2.8. *Let $\mathcal{A}$, $\mathcal{B}$ be as above, $1 < p,q < \infty$. For $v \in \mathrm{C}_c^\infty(\Omega,\mathbb{V})$, we have the decomposition*

$$v = \mathcal{B}u + \mathcal{A}^*w, \quad \textit{with } \|D^k u\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant c\|v\|_{\mathrm{L}^p(\Omega)},\ \|D^\ell w\|_{\mathrm{L}^q(\mathbb{R}^n)} \leqslant c\|v\|_{\mathrm{W}^{-\ell,q}(\Omega)},$$

where

$$\hat{u}(\xi) := \mathcal{B}^\dagger(\xi)\hat{v}(\xi), \quad w := (\mathcal{A}^*)^\dagger(\xi)\hat{v}(\xi).$$

2.3. **$\mathcal{A}$-quasiconvex integrands.** A known necessary condition for lower semi-continuity in the topology we are working with is that of $\mathcal{A}$-quasiconvexity (see [17]): we say that a locally bounded Borel function $f\colon \mathbb{V} \to \mathbb{R}$ is $\mathcal{A}$-quasiconvex if

$$f(z) \leqslant \fint_{\mathbb{T}^n} f(z+v(x))\mathrm{d}x, \quad \text{for all } v \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}), \text{ s.t. } \mathcal{A}v = 0,\ \fint_{\mathbb{T}^n} v(x)\mathrm{d}x = 0.$$

We say that a non-autonomous normal integrand $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ is $\mathcal{A}$-quasiconvex if $F(x_0,\bullet)$ is $\mathcal{A}$-quasiconvex for $\mathscr{L}^n$-a.e. $x_0 \in \Omega$. It is well known that such an $\mathcal{A}$-quasiconvex integrand f is convex in the directions of the wave cone $\Lambda_\mathcal{A}$ (see (1.4) and [17, Prop. 3.4]). In particular, if the wave cone is spanning $\mathbb{V}$, one can argue that f is locally Lipschitz. In particular, in this case, the definition (1.8) of the p-upper recession function can be shown to equal

$$f_p^\infty(z) = \limsup_{t\to\infty} \frac{f(tz)}{t^p} \quad \text{for } z \in \mathbb{V}.$$

This fact will be used without mention. Nevertheless, the limit superior in the formula cannot be replaced with the limit, see [31]. It is no surprise then, that we will need the following approximation from above of quasiconvex integrands by quasiconvex integrands that have regular recession function.

Lemma 2.9. *Let $f\colon \mathbb{V} \to \mathbb{R}$ be $\mathcal{A}$-quasiconvex. Then there exist $\mathcal{A}$-quasiconvex $\Phi \in \mathbb{E}_p$ such that*

$$\Phi_i \downarrow f \quad \text{and} \quad \Phi_{i,p}^\infty \downarrow f_p^\infty$$

as $i \to \infty$ pointwisely in $\mathbb{V}$.

A proof of this fact can be given by modifying the argument in [23, Lem. 6.3]. Moreover, the Φ_i can be chosen to be p-homogeneous outside a large ball that increases with i.

Even though this fact is only used implicitly in the sequel (as part of the omitted proof of Proposition 5.1), it was shown in [34] that, in the presence of (1.3) and without (1.4), an integrand f is $\mathcal{A}$-quasiconvex if and only if

$$f(z) \leqslant \fint_\Omega f(z + \mathcal{B}u(x))\mathrm{d}x, \quad \text{for all } u \in \mathrm{C}_c^\infty(\Omega, \mathbb{U}).$$

Together with the spanning cone condition (1.4), this has been used in [26, Lem. 4.7] to reduce $\mathcal{A}$-quasiconvexity to k-quasiconvexity.

3. The decomposition lemmas

Lemma 3.1 (Decomposition lemma, anisotropic version)**.** *Let $\mathcal{A}$ as in* (1.2) *be a constant rank operator with potential operator $\mathcal{B}$ such that* (1.5) *holds. Let $1 < p, q < \infty$ and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \to \mathcal{A}v \text{ in } \mathrm{W}^{-\ell,q}(\Omega, \mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu}$. Then there exist sequences $(u_j) \subset \mathrm{C}_c^\infty(\Omega, \mathbb{U})$ and $(b_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ such that

$$\begin{aligned} &v_j = v + \mathcal{B}u_j + b_j, \\ &\mathcal{B}u_j,\ b_j \rightharpoonup 0 \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}), \\ &(D^k u_j) \text{ is } p\text{-uniformly integrable}, \\ &b_j \to 0 \text{ in } \mathscr{L}^n\text{-measure}. \end{aligned}$$

Therefore, we have that, in $\mathrm{Y}^p(\Omega, \mathbb{V})$

$$(v + \mathcal{B}u_j) \text{ generates } ((\nu_x)_{x\in\Omega}, 0, \mathtt{n/a}) \quad \text{and} \quad (b_j) \text{ generates } \left((\delta_0)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\bar\Omega}\right).$$

Proof. Define the truncation maps

$$\mathcal{T}_\alpha(z) := \begin{cases} z & |z| \leqslant \alpha \\ k\frac{z}{|z|} & |z| > \alpha \end{cases},$$

which can be used to see that

$$\lim_{\alpha\to\infty} \lim_{j\to\infty} \int_\Omega |\mathcal{T}_\alpha v_j|^p = \int_\Omega \langle \nu_x, |\,\boldsymbol{\cdot}\,|^p \rangle \mathrm{d}x,$$

so that we can employ a diagonalization argument to see that there exists a sequence $\alpha_j \uparrow \infty$ such that

$$\lim_{j\to\infty} \int_\Omega |\mathcal{T}_{\alpha_j} v_j|^p = \int_\Omega \langle \nu_x, |\,\boldsymbol{\cdot}\,|^p \rangle \mathrm{d}x.$$

By use of Theorem 2.6 applied to the integrand $|\,\boldsymbol{\cdot}\,|^p$ and the sequence $(\mathcal{T}_{\alpha_j} v_j)$, we see that the sequence is p-uniformly integrable. Since (v_j) converges weakly in L^1, it is uniformly integrable, so that, for $\varepsilon > 0$,

$$\varepsilon \mathscr{L}^n(|v_j - \mathcal{T}_{\alpha_j} v_j| > \varepsilon) \leqslant \int_{|v_j| > \alpha_j} |v_j| \left(1 - \frac{\alpha_j}{|v_j|}\right) \mathrm{d}x \leqslant \int_{|v_j|>\alpha_j} |v_j| \mathrm{d}x \to 0,$$

so that $(v_j - \mathcal{T}_{\alpha_j} v_j)$ converges to zero in measure.

We write $\boldsymbol{\nu}_1 := ((\nu_x)_{x\in\Omega},\, 0,\, \texttt{n/a})$ and $\boldsymbol{\nu}_2 := \big((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega}\big)$ and conclude from Theorems 2.5 and 2.6 that $(\mathcal{T}_{\alpha_j} v_j)$ generates $\boldsymbol{\nu}_1 \in \mathrm{Y}^p(\Omega,\mathbb{V})$ and $(v_j - \mathcal{T}_{\alpha_j} v_j)$ generates $\boldsymbol{\nu}_2 \in \mathrm{Y}^p(\Omega,\mathbb{V})$.

Let now $r>1$ be a number such that $r<p$ and $r\leqslant q$. We claim that $\mathcal{T}_{\alpha_j} v_j - v_j \to 0$ in $\mathrm{L}^r(\Omega,\mathbb{V})$. To see this, write

$$\|\mathcal{T}_{\alpha_j} v_j - v_j\|^r_{\mathrm{L}^r(\Omega)} \leqslant c\int_{|v_j|\geqslant\alpha_j} |v_j|^r \mathrm{d}x \leqslant c\int_{|v_j|\geqslant\alpha_j} \frac{|v_j|^p}{\alpha_j^{p-r}}\mathrm{d}x \leqslant \frac{c}{\alpha_j^{p-r}}\int_\Omega |v_j|^p\mathrm{d}x \to 0.$$

In particular, $\mathcal{A}\mathcal{T}_{\alpha_j} v_j \to \mathcal{A} v$ in $\mathrm{W}^{-\ell,r}(\Omega,\mathbb{W})$.

We now aim to find a sequence of cut off functions $\rho_j \in \mathrm{C}^\infty_c(\Omega,[0,1])$ such that $\rho_j \uparrow 1$ that makes $\mathcal{A}\big(\rho_j(\mathcal{T}_{\alpha_j} v_j - v)\big)$ well behaved. First, note that for any such sequence we have that $(\rho_j \mathcal{T}_{\alpha_j} v_j)$ is p-uniformly integrable and that $\big((1-\rho_j)(\mathcal{T}_{\alpha_j} v_j - v)\big)$ converges in measure to zero.

$$v_j := \big[\rho_j(\mathcal{T}_{\alpha_j} v_j - v) + v\big] + \big[(v_j - \mathcal{T}_{\alpha_j} v_j) + (1-\rho_j)(\mathcal{T}_{\alpha_j} v_j - v)\big]\,,$$

where the first term converges weakly in L^p to v and generates $\boldsymbol{\nu}_1$ and the second term converges in measure to zero and generates $\boldsymbol{\nu}_2$. It remains to preserve the differential structure, which adds restrictions to (ρ_j). We write $\tilde v_j = \mathcal{T}_{\alpha_j} v_j - v$, so that $\mathcal{A}\tilde v_j \to 0$ in $\mathrm{W}^{-\ell,r}(\Omega,\mathbb{W})$ and

$$\mathcal{A}(\rho_j \tilde v_j) = \rho_j \mathcal{A}\tilde v_j + \sum_{|\alpha|=\ell}\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}\rho_j A_\alpha \partial^\beta \tilde v_j, \tag{3.1}$$

where $\partial^\beta \tilde v_j \to 0$ in $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{W})$ by the compact Sobolev embedding. Choosing the derivatives of (ρ_j) to blow up slow enough near the boundary of Ω, we can obtain that $\mathcal{A}(\rho_j\tilde v_j) \to 0$ in $\mathrm{W}^{-\ell,r}(\Omega,\mathbb{W})$. Since ρ_j is compactly supported inside Ω, we can mollify and assume that $\rho_j \equiv 1$ and $\tilde v_j \in \mathrm{C}^\infty_c(\Omega,\mathbb{V})$, which we identify with their extension by zero to $\mathbb{R}^n$ without mention. With this new notation, we record that $(v+\tilde v_j)$ generates $\boldsymbol{\nu}_1$ and $(v_j - v - \tilde v_j)$ generates $\boldsymbol{\nu}_2$.

We can define, cf. Proposition 2.8,

$$\hat u_j(\xi) := \mathcal{B}^\dagger(\xi)\mathscr{F}\tilde v_j(\xi), \quad \text{so that } \widehat{D^k u_j}(\xi) = \mathcal{B}^\dagger(\xi)\mathscr{F}\tilde v_j(\xi)\otimes\xi^{\otimes k} =: \mathscr{F}[H\tilde v_j](\xi) \quad \text{for } \xi\neq 0.$$

We can then infer that $\tilde v_j - \mathcal{B}u_j \to 0$ in $\mathrm{L}^r(\mathbb{R}^n,\mathbb{V})$, so that $(v_j - v - \mathcal{B}u_j)$ generates $\boldsymbol{\nu}_2$. We claim that $(D^k u_j)$ is p-uniformly integrable in Ω. In that case, we retrieve $\mathcal{B}u_j = \mathbf{T}(D^k u_j)$, where the tensor $\mathbf{T}$ is the linear map in (1.5). It will follow that $(v+\mathcal{B}u_j)$ generates $\boldsymbol{\nu}_1$.

To prove this, first note that for $\alpha>0$, by the Hörmander–Mikhlin multiplier theorem, we have

$$\sup_j \int_{\mathbb{R}^n} |H\tilde v_j - H\mathcal{T}_\alpha \tilde v_j|^p\mathrm{d}x \leqslant c\sup_j\int_{\mathbb{R}^n}|\tilde v_j - \mathcal{T}_\alpha v_j|^p\mathrm{d}x \to 0 \quad \text{as } \alpha\to\infty$$

by p-uniform integrability of $(\tilde v_j)$. Let $\varepsilon>0$ and choose $\alpha>0$ such that the right hand side is less than ε. Let $s>p$ and notice that, again by the Hörmander–Mikhlin multiplier theorem, we have

$$\|H\mathcal{T}_\alpha\tilde v_j\|_{\mathrm{L}^s(\mathbb{R}^n)} \leqslant c\|\mathcal{T}_\alpha \tilde v_j\|_{\mathrm{L}^s(\Omega)} \leqslant c\alpha,$$

so that $(H\mathcal{T}_\alpha\tilde v_j)$ is p-uniformly integrable. Then there exists $\delta>0$ such that $\mathscr{L}^n(E)<\varepsilon$ implies that

$$\int_E |H\mathcal{T}_\alpha \tilde v_j|^p\mathrm{d}x \leqslant \varepsilon.$$

We can therefore estimate

$$\int_E |D^k u_j|^p\mathrm{d}x = \int_E |H\tilde v_j|^p\mathrm{d}x \leqslant c\int_{\mathbb{R}^n}|H\tilde v_j - H\mathcal{T}_\alpha\tilde v_j|^p\mathrm{d}x + c\int_E |H\mathcal{T}_\alpha\tilde v_j|^p\mathrm{d}x \leqslant c\varepsilon,$$

which concludes the proof of the claim that $(\mathcal{B}u_j)$ is uniformly integrable.

It remains to use cut off functions to prove that we can assume that $\mathcal{B}u_j$ are compactly supported inside Ω. To this end, let $\phi_j \in \mathrm{C}_c^\infty(\Omega, [0,1])$ be such that $\phi_j \uparrow 1$ be such that $\mathcal{B}(\rho_j u_j)$ is well behaved in a sense that we now describe. First, note that since $\partial^\beta u_j \to 0$ in $\mathrm{L}^p(\mathbb{R}^n, \mathrm{SLin}^k(\mathbb{R}^n, \mathbb{V}))$, we have by the compact Sobolev embedding that $\partial^\beta u_j \to 0$ in $\mathrm{L}^p(\Omega, \mathbb{V})$ for $|\beta| < k$. In particular, by a Leibniz rule computation similar to the one in (3.1), we can choose ϕ_j to be controlled in $\mathrm{C}^k(\bar{\Omega})$ such that $\mathcal{B}(\phi_j u_j) - \phi_j \mathcal{B}u_j \to 0$ in $\mathrm{L}^p(\Omega, \mathbb{W})$. In particular, $(\mathcal{B}(\phi_j u_j))$ is p-uniformly integrable and $\mathcal{B}(\phi_j u_j) - \mathcal{B}u_j \to 0$ in measure in Ω.

It follows that we can assume that $u_j \in \mathrm{C}_c^\infty(\Omega, \mathbb{U})$ and we can set $b_j := v_j - v - \mathcal{B}u_j$ so that all the required properties are satisfied. □

Proof of the Decomposition Lemma 1.1. Using the Decomposition Lemma 3.1 with $p = q$, we can write $v_j = v + \mathcal{B}u_j + b_j$ with (u_j) as required and (b_j) generating

$$\left((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar{\Omega}}\right).$$

Consequently, we have that $\mathcal{A}b_j \to 0$ in $\mathrm{W}^{-\ell,p}(\Omega, \mathbb{V})$. Selecting cut-off (test) functions $0 \leqslant \rho_j \uparrow 1$. Proceeding like in the proof of Lemma 3.1, we can ensure that $\mathcal{A}(\rho_j b_j) \to 0$ in $\mathrm{W}^{-\ell,p}(\Omega, \mathbb{V})$. Therefore, the same is true of $(\mathcal{A}((1-\rho_j)b_j))$, and clearly $(\rho_j b_j)$, $((1-\rho_j)b_j)$ both converge to zero in measure and weakly in L^p. Since all $(1-\rho_j)b_j = b_j$ near $\partial\Omega$, it is easy to see that $((1-\rho_j)b_j)$ generates $((\delta_0)_{x\in\Omega},\, \lambda \llcorner \partial\Omega,\, (\nu_x^\infty)_{x\in\partial\Omega})$.

Next, one can use the same Helmholtz decomposition as in Lemma 3.1 to split

$$\rho_j b_j = \mathcal{B}U_j + \mathcal{A}^* w_j,$$

which, in this case are such that $\mathcal{A}^* w_j \to 0$ in $\mathrm{L}^p(\mathbb{R}^n, \mathbb{V})$. Repeating the cut-off function argument at the end of the proof of Lemma 3.1, we can define $\tilde{u}_j := \phi_j U_j$ in a way such that $\mathcal{B}\left((1-\phi_j)U_j\right) \to 0$ in $\mathrm{L}^p(\Omega, \mathbb{V})$. We can then conclude that $\tilde{u}_j$ thus defined and $\tilde{b}_j := (1-\rho_j)b_j + \mathcal{A}^* w_j + \mathcal{B}\left((1-\phi_j)U_j\right)$ satisfy the conditions of the lemma. □

4. The Jensen inequalities and the lower semi-continuity theorem

We begin this section by proving the Jensen inequalities (1.9). To this end, we state without proof a so-called localization result:

Proposition 4.1. *Let $\mathcal{A}$ as in* (1.2) *be a constant rank operator with potential operator $\mathcal{B}$ such that* (1.5) *holds. Let $1 < p, q < \infty$ and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \to \mathcal{A}v \text{ in } \mathrm{W}^{-\ell,q}(\Omega, \mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu} = \left((\nu_x)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar{\Omega}}\right)$. We write $\lambda = \lambda^a(x)\mathscr{L}^n \llcorner \Omega + \lambda^s$ for the Radon–Nýkodim decomposition of λ. We have that

(a) *For $\mathscr{L}^n$-a.e. $x_0 \in \Omega$, we have that the homogeneous p-Young measure*

$$\left((\nu_{x_0})_{y\in Q_1(0)},\, 0,\, \texttt{n/a}\right) \text{ is generated by } (v(x_0) + \mathcal{B}u_j),$$

where $u_j \in \mathrm{C}_c^\infty(Q_1(0), \mathbb{U})$.

Suppose that $p = q$.

(b) *For $\mathscr{L}^n$-a.e. $x_0 \in \Omega$, we have that the homogeneous p-Young measure*

$$\left((\delta_0)_{x_0\in Q_1(0)},\, \lambda^a(x_0)\mathscr{L}^n \llcorner Q_1(0),\, (\nu_{x_0}^\infty)_{y\in Q_1(0)}\right) \text{ is generated by } (\mathcal{B}\tilde{u}_j),$$

where $\tilde{u}_j \in \mathrm{C}_c^\infty(Q_1(0), \mathbb{U})$.

(c) *For λ^s-a.e. $x_0 \in \Omega$, there exists a (tangent measure of λ^s at x_0) $\tau \in \mathcal{M}_1^+(\bar{Q}_1(0))$, such that we have that the homogeneous p-Young measure*

$$\left((\delta_0)_{x_0\in Q_1(0)},\, \tau,\, (\nu_{x_0}^\infty)_{y\in\bar{Q}_1(0)}\right) \text{ is generated by } (\mathcal{B}U_j),$$

where $U_j \in \mathrm{C}_c^\infty(Q_1(0), \mathbb{U})$.

A proof can be found in [26, Sec. 5], see also [3]. Using this result, we can proceed to prove the Jensen inequalities (1.9). Since the oscillation inequality holds for anisotropic constraints, we will split the proof in two lemmas.

Lemma 4.2. *Suppose that $\mathcal{A}$ satisfies conditions* (1.3) *and* (1.4)*. Let $1<p,q<\infty$ and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega,\mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \to \mathcal{A}v \text{ in } \mathrm{W}^{-\ell,q}(\Omega,\mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu} = \big((\nu_x)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega}\big)$. Then for all $\mathcal{A}$-quasiconvex $f\colon \mathbb{V}\to\mathbb{R}$ satisfying (1.6)*, we have that*

$$\langle f,\nu_x\rangle \geqslant f(\bar\nu_x) \quad \text{for } \mathscr{L}^n\text{-a.e. } x\in\Omega.$$

Lemma 4.3. *Suppose that $\mathcal{A}$ satisfies conditions* (1.3) *and* (1.4)*. Let $1<p<\infty$ and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega,\mathbb{V}) \quad \text{with} \quad \mathcal{A}v_j \to \mathcal{A}v \text{ in } \mathrm{W}^{-\ell,p}(\Omega,\mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu} = \big((\nu_x)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega}\big)$. Then for all $\mathcal{A}$-quasiconvex $f\colon \mathbb{V}\to\mathbb{R}$ satisfying (1.6)*, we have that*

$$\langle f_p^\infty,\nu_x^\infty\rangle \geqslant 0 \quad \text{for } \lambda\text{-a.e. } x\in\Omega.$$

Proof of Lemma 4.2. Let $x_0\in\Omega$ be in a set of full Lebesgue measure where $\bar\nu_{x_0}=v(x_0)$ (a Lebesgue point of v) and Proposition 4.1(a) applies. We have that, for autonomous and $\mathcal{A}$-quasiconvex $\Phi\in\mathbb{E}_p$

$$\langle \Phi,\nu_{x_0}\rangle = \lim_{j\to\infty}\int_{Q_1(0)} \Phi(v(x_0)+\mathcal{B}u_j(y))\mathrm{d}y \geqslant \Phi(v(x_0)).$$

If f is autonomous and $\mathcal{A}$-quasiconvex but does not necessarily posses a strong recession function, we employ Lemma 2.9 to approximate f from above with well behaved Φ. The conclusion follows from the monotone convergence theorem. □

Proof of Lemma 4.3. Let now $x_0\in\Omega$ be $\mathscr{L}^n$-significant such that Proposition 4.1(b) applies. Letting a well behaved $\mathcal{A}$-quasiconvex integrand Φ approximate f as in the proof of Lemma 4.2, we note that $\Phi_p^\infty\in\mathbb{E}_p$ is also $\mathcal{A}$-quasiconvex, so we can write

$$\lambda^a(x_0)\langle \Phi_p^\infty,\nu_x^\infty\rangle = \lim_{j\to\infty}\int_{Q_1(0)} \Phi_p^\infty(\mathcal{B}\tilde u_j(y))\mathrm{d}y \geqslant \Phi_p^\infty(0)=0,$$

which implies the required inequality since $\lambda\in\mathcal{M}^+(\bar\Omega)$.

If we now look at a λ^s-significant $x_0\in\Omega$ such that Proposition 4.1(c) applies, we have, in a similar fashion,

$$\tau(\bar Q_1(0))\langle \Phi_p^\infty,\nu_x^\infty\rangle = \lim_{j\to\infty}\int_{Q_1(0)} \Phi_p^\infty(\mathcal{B}U_j(y))\mathrm{d}y \geqslant \Phi_p^\infty(0)=0,$$

and we can conclude as before since τ is a probability measure. □

We conclude this section with a proof of the lower semi-continuity theorem:

Proof of Theorem 1.2. By Proposition 2.7 we have that

$$\liminf_{j\to\infty}\int_\Omega F(x,v_j(x))\mathrm{d}x \geqslant \int_\Omega \langle \nu_x, F(x,\bullet)\rangle \mathrm{d}x \geqslant \int_\Omega F(x,\bar\nu_x)\mathrm{d}x = \int_\Omega F(x,v(x))\mathrm{d}x$$

where the second inequality follows from Lemma 4.2. □

5. Proof of the characterization result

It remains to prove Theorem 1.3. The Jensen inequalities were already proved in Lemmas 4.2 and 4.3. To establish the converse, we proceed with adapting the strategy from [25, Sec. 3]. We begin with the case of homogeneous Young measures.

Let $Q \subset \mathbb{R}^n$ be a cube and $z \in \mathbb{V}$ and define

$$\mathrm{Y}^p_h(z) := \Big\{(\nu^0, \nu^\infty) \in \mathcal{M}^+_1(\mathbb{V}) \times \mathcal{M}^+(S_{\mathbb{V}})\colon \text{there exist } u_j \in \mathrm{C}^\infty_c(Q, \mathbb{U}) \text{ s.t. for all } \Phi \in \mathbb{E}_{p,a}$$
$$\lim_{j\to\infty} \fint_Q \Phi(z + \mathcal{B}u_j(x))\mathrm{d}x = \langle \nu^0, \Phi\rangle + \langle \nu^\infty, \Phi^\infty_p\rangle\Big\},$$

where $\mathbb{E}_{p,a}(\mathbb{V})$ denotes the set of autonomous integrands in $\mathbb{E}_p(\Omega, \mathbb{V})$, i.e.,

$$\mathbb{E}_{p,a}(\mathbb{V}) = \left\{\Phi \in \mathrm{C}(\mathbb{V})\colon \Phi^\infty_p(x, z) := \lim_{t\to\infty} \frac{\Phi(tz)}{t^p} \in \mathbb{R} \text{ locally uniformly for } z \in \mathbb{V}\right\}.$$

It is easy to see that, with the norm induced from $\mathbb{E}_p(\Omega, \mathbb{V})$, we have that $\mathrm{Y}^p_h(z) \subset \mathbb{E}_{p,a}(\mathbb{V})^* \simeq \mathcal{M}(\mathbb{V}) \times \mathcal{M}(S_{\mathbb{V}})$, where the isomorphism is given by the map T_p. We record that, since $p > 1$, we have that $\bar{\nu}^0 = z$ for elements of $\mathrm{Y}^p_h(z)$. Finally, let us mention that in the "inhomogenization" argument we will only look at measures $(\nu^0, \nu^\infty) \in \mathrm{Y}^p_h$ that have $\nu^0 = \delta_0$ or $\nu^\infty \equiv 0$, which is completely unlike in [25].

We can now formulate the homogeneous step of the converse of Theorem 1.3:

Proposition 5.1. *Suppose that $\mathcal{A}$ satisfies conditions* (1.3) *and* (1.4)*. Let $\mathcal{B}$ be a potential operator for $\mathcal{A}$ such that* (1.5) *holds. Let $\nu := (\nu^0, \nu^\infty) \in \mathcal{M}^+_1(\mathbb{V}) \times \mathcal{M}^+(S_{\mathbb{V}})$ and $z \in \mathbb{V}$. Then $\nu \in \mathrm{Y}^p_h(z)$ if and only if $\bar{\nu}^0 = z$ and*

$$\langle \nu^0, f\rangle + \langle \nu^\infty, f^\infty_p\rangle \geqslant f(z) \quad \textit{for all } \mathcal{A}\textit{-quasiconvex } f\colon \mathbb{V} \to \mathbb{R} \textit{ satisfying } (1.6).$$

The proof follows the lines of the argument in [25, Sec. 3.3] exactly. We can now proceed with the proof of the main result, which follows closely the construction in [25, Sec. 3.4], see also [33, Sec. 3.3.3]:

Proof of Theorem 1.3. We already explained that we need only prove the converse. To this end, let $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ be such that $\lambda(\partial\Omega) = 0$. By Theorem 2.5 and 2.6, we have that it suffices to show that there exist sequences (u_j), $(\tilde{u}_j) \subset \mathrm{C}^\infty_c(\Omega, \mathbb{U})$ such that

$$(v + \mathcal{B}u_j) \text{ generates } ((\nu_x)_{x\in\Omega}, 0, \texttt{n/a}) \quad \text{and} \quad (\mathcal{B}\tilde{u}_j) \text{ generates } ((\delta_0)_{x\in\Omega}, \lambda, (\nu^\infty_x)_{x\in\Omega}).$$

Indeed, this is enough since we would have that $(\mathcal{B}u_j)$ is p-uniformly integrable and $\mathcal{B}\tilde{u}_j \to 0$ in measure, while both sequences converge weakly to 0 in $\mathrm{L}^p(\Omega, \mathbb{V})$. In this case, one can apply the Decomposition Lemma 1.1 to refine the two sequences.

We will test with integrands $\varphi \otimes \Phi \in \mathrm{C}(\bar{\Omega}) \times \mathbb{E}_{p,a}(\mathbb{V})$ as given by Lemma 2.4. In particular, we can assume that $\|\varphi\|_{\mathrm{Lip}}, \|T_p\Phi\|_{\mathrm{Lip}} \leqslant 1$, so that

$$|\Phi(z) - \Phi(z')| \leqslant c|z - z'|(1 + |z| + |z'|)^{p-1} \quad \text{for } z, z' \in \mathbb{V}.$$

We are working with $\|\bullet\|_{\mathrm{Lip}} := \|\bullet\|_{\mathrm{L}^\infty} + \|D\bullet\|_{\mathrm{L}^\infty}$. Let $\varepsilon > 0$.

We write $g(x) := \varphi(x)\langle \nu_x, \Phi\rangle$ and $g_0(x) := \langle \nu_x, \Phi_0\rangle$ where $\Phi_0 = (1 + |\bullet|)^p$, so $g, g_0 \in \mathrm{L}^1(\Omega)$ by the moment condition. We apply Lusin's theorem in the following way: There exists a compact set $C \subset \Omega$ such that, with $G = (g, g_0)$,

$$\mathscr{L}^n(\Omega \setminus C) < \varepsilon|\Omega|, \quad \int_{\Omega\setminus C} |G|\mathrm{d}x < \varepsilon|\Omega|, \quad \text{and } G\big|_C \text{ is continuous.}$$

Using Tietze's extension theorem, we can find $\tilde{G} =: (\tilde{g}, \tilde{g}_0) \in \mathrm{C}(\bar{\Omega})$ such that $\tilde{G} = G$ in C and $\|\tilde{G}\|_{\mathrm{L}^\infty(\Omega)} = \|G\|_{\mathrm{L}^\infty(C)}$. Moreover, $\tilde{G}$ is uniformly continuous, so we can find $\delta \in (0, \varepsilon)$ such that $|\tilde{G}(x) - \tilde{G}(x')| < \varepsilon$ whenever $|x - x'| < \delta$. Finally, consider a regular grid of cubes in $\mathbb{R}^n$ of side length $\delta/2$; we write $\mathcal{F}_\delta$ for the family of such cubes that are contained in Ω. Since $\mathscr{L}^n(\partial\Omega)=0$, it is clear that $|\bigcup\mathcal{F}_\delta|\uparrow|\Omega|$ as $\delta\downarrow 0$. We write

$$\mathcal{F}_\delta^o := \{Q\in\mathcal{F}_\delta : Q\cap C\neq\emptyset\}.$$

Then $\mathcal{F}_\delta^o$ covers $C\cap\bigcup\mathcal{F}_\delta$, so that we can assume by taking δ smaller that

$$\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^o}|G|\mathrm{d}x<\varepsilon|\Omega|.$$

For each cube $Q\in\mathcal{F}_\delta^o$, we choose an arbitrary $x_Q\in Q\cap C$ that are Lebesgue points of the barycentre $v(x)=\bar{\nu}_x$ and such that the oscillation Jensen inequality holds at x_Q. We also record that

$$|G(x_Q)-\tilde{G}(x)|<\varepsilon\quad\text{for all } x\in Q.$$

We can also assume that we have a piecewise constant approximation of the barycentre in $\mathrm{L}^p(\Omega,\mathbb{V})$

$$\int_\Omega|v-v^\varepsilon|^p\mathrm{d}x\leqslant\varepsilon|\Omega|,\quad\text{where } v^\varepsilon:=\sum_{Q\in\mathcal{F}_\delta^o}v(x_Q)\mathbf{1}_Q.$$

As a consequence of Proposition 5.1 with $\nu^0=\nu_{x_Q}$, $\nu^\infty=0$ we can find $u_Q^\varepsilon\in\mathrm{C}_c^\infty(Q,\mathbb{U})$ such that

$$\left|\langle\nu_{x_Q},\Phi\rangle-\fint_Q\Phi(\bar{\nu}_{x_Q}+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x\right|+\left|\langle\nu_{x_Q},\Phi_0\rangle-\fint_Q\Phi_0(\bar{\nu}_{x_Q}+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x\right|<\varepsilon.$$

Recall here that $\Phi_0=(1+|\bullet|)^p$. We can begin to estimate

$$\left|\int_\Omega g\mathrm{d}x-\int_{\bigcup\mathcal{F}_\delta^0}\tilde{g}\mathrm{d}x\right|\leqslant\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^o}|g|\mathrm{d}x+\int_{F_\delta^o}|g-\tilde{g}|\mathrm{d}x\leqslant 2\varepsilon|\Omega|,$$

so that

$$\left|\int_{\bigcup\mathcal{F}_\delta^0}\tilde{g}\mathrm{d}x-\sum_{Q\in\mathcal{F}_\delta^o}|Q|g(x_Q)\right|\leqslant\varepsilon|\Omega|.$$

We can estimate further

$$\left|\sum_{Q\in\mathcal{F}_\delta^o}\left(|Q|g(x_Q)-\varphi(x_Q)\int_Q\Phi(\bar{\nu}_{x_Q}+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x\right)\right|\leqslant\varepsilon|\Omega|.$$

We then have that

$$\sum_{Q\in\mathcal{F}_\delta^o}\varphi(x_Q)\int_Q\Phi(\bar{\nu}_{x_Q}+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x=\sum_{Q\in\mathcal{F}_\delta^o}\int_Q\varphi(x)\Phi(\bar{\nu}_{x_Q}+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x+\mathcal{E}_1,$$

where

$$\begin{aligned}|\mathcal{E}_1|&\leqslant c\sum_{Q\in\mathcal{F}_\delta^o}\int_Q|\varphi(x_Q)-\varphi(x)|\Phi_0(\bar{\nu}_{x_Q}+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x\leqslant c\delta\sum_{Q\in\mathcal{F}_\delta^o}|Q|(\langle\nu_{x_Q},\Phi_0\rangle+\varepsilon)\\&\leqslant c\delta\sum_{Q\in\mathcal{F}_\delta^o}\left(\int_Q\langle\nu_x,\Phi_0\rangle\mathrm{d}x+2\varepsilon|Q|\right)\leqslant c\delta\left(\int_\Omega\langle\nu_x,\Phi_0\rangle\mathrm{d}x+2\varepsilon|\Omega|\right),\end{aligned}\tag{5.1}$$

where the integral is finite by the moment condition. We make and recall the abbreviations

$$u^\varepsilon:=\sum_{Q\in\mathcal{F}_\delta^o}u_Q^\varepsilon\in\mathrm{C}_c^\infty(\Omega,\mathbb{U})\quad\text{and}\quad v^\varepsilon=\sum_{Q\in\mathcal{F}_\delta^o}v(x_Q)\mathbf{1}_Q\in\mathrm{L}^p(\Omega,\mathbb{V}).$$

We next look at

$$\sum_{Q\in\mathcal{F}_\delta^o}\int_Q \varphi(x)\Phi(v(x_Q)+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x = \sum_{Q\in\mathcal{F}_\delta^o}\int_Q \varphi(x)\Phi(v(x)+\mathcal{B}u_Q^\varepsilon(x))\mathrm{d}x + \mathcal{E}_2,$$

where, by using $\|\varphi\|_{\mathrm{L}^\infty}\leqslant 1$,

$$\begin{aligned}|\mathcal{E}_2| &\leqslant c\int_\Omega |v-v^\varepsilon|\,(1+|v+\mathcal{B}u^\varepsilon|+|v^\varepsilon+\mathcal{B}u^\varepsilon|)^{p-1}\,\mathrm{d}x\\ &\leqslant c\|v-v^\varepsilon\|_{\mathrm{L}^p(\Omega)}\|1+|v+\mathcal{B}u^\varepsilon|+|v^\varepsilon+\mathcal{B}u^\varepsilon|\|^{p-1}_{\mathrm{L}^p(\Omega)}\\ &\leqslant c\|v-v^\varepsilon\|_{\mathrm{L}^p(\Omega)}\left(\left(\int_\Omega \Phi_0(v^\varepsilon+\mathcal{B}u^\varepsilon)\mathrm{d}x\right)^{(p-1)/p}+\|v-v^\varepsilon\|^{p-1}_{\mathrm{L}^p(\Omega)}\right)\end{aligned}$$

Since $\|v-v^\varepsilon\|_{\mathrm{L}^p(\Omega)}\leqslant(\varepsilon|\Omega|)^{1/p}$ and the estimation

$$\int_\Omega \Phi_0(v^\varepsilon+\mathcal{B}u^\varepsilon)\mathrm{d}x\leqslant c\left(\int_\Omega\langle\nu_x,\Phi_0\rangle\mathrm{d}x+\varepsilon|\Omega|\right)$$

from (5.1), we are very close to conclude. Writing

$$\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^o}\varphi(x)\Phi(v(x)+\mathcal{B}u^\varepsilon(x))\mathrm{d}x\leqslant c\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^o}(1+|v|)^p\mathrm{d}x\leqslant c\varepsilon|\Omega|$$

and collecting estimates, we have that

$$\left|\int_\Omega\varphi(x)\langle\nu_x,\Phi\rangle\mathrm{d}x-\int_\Omega\varphi(x)\Phi(v(x)+\mathcal{B}u^\varepsilon(x))\mathrm{d}x\right|\to 0\quad\text{as }\varepsilon\downarrow 0.$$

We have thus showed that the oscillation measure has the right gradient structure.

We carry on with the concentration part. We now consider the functions $h(x):=\varphi(x)\langle\nu_x^\infty,\Phi_p^\infty\rangle$ and $h_0(x):=\langle\nu_x^\infty,\Phi_{0,p}^\infty\rangle$, where we recall that $\Phi_0=(1+|\boldsymbol{\cdot}|)^p$. We have that $H:=(h,h_0)\in\mathrm{L}^1(\Omega;\mathrm{d}\lambda)$. As in the previous case, we will apply Lusin's theorem and Tietze's extension theorem to find a compact set $K\subset\Omega$ and a continuous extension $\tilde{H}=:(\tilde{h},\tilde{h}_0)\in\mathrm{C}(\bar{\Omega})$, such that

$$\lambda(\Omega\setminus K)<\varepsilon\lambda(\Omega),\qquad\int_{\Omega\setminus K}|H|\mathrm{d}\lambda<\varepsilon\lambda(\Omega),\qquad\tilde{H}\big|_K=H,\qquad\|\tilde{H}\|_{\mathrm{L}^\infty(\Omega)}=\|H\|_{\mathrm{L}^\infty(K)}.$$

We then choose $\delta\in(0,\varepsilon)$ and a collection $\mathcal{F}_\delta^c$ of cubes in *exact* analogy with the case of the oscillation measures, by replacing $\mathscr{L}^n$ with λ. At this stage we also use the assumption $\lambda(\partial\Omega)=0$. Finally, for each $Q\in\mathcal{F}_\delta^c$, we choose $x_Q\in Q\cap K$ such that the concentration Jensen inequality holds at x_Q.

By Proposition 5.1 with $\nu^0=\delta_0$ and $\nu^\infty=|Q|^{-1}\lambda(Q)\nu_{x_Q}^\infty$ we can find $\tilde{u}_Q^\varepsilon\in\mathrm{C}_c^\infty(Q,\mathbb{U})$ such that

$$\begin{aligned}&\left|\Phi(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_p^\infty\rangle-\fint_Q\Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x\right|<\varepsilon,\\ &\left|\Phi_0(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_{0,p}^\infty\rangle-\fint_Q\Phi_0(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x\right|<\varepsilon.\end{aligned}$$

We can then estimate

$$\begin{aligned}\int_\Omega\varphi\mathrm{d}x\Phi(0)+\int_\Omega h\mathrm{d}\lambda=\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^c}\varphi\mathrm{d}x\Phi(0)+\sum_{Q\in\mathcal{F}_\delta^c}|Q|\varphi(x_Q)\left(\Phi(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_p^\infty\rangle\right)\\ +\mathcal{E}_3,\end{aligned}$$

where

$$|\mathcal{E}_3| \leqslant \sum_{Q\in\mathcal{F}_\delta^c} \int_Q |\varphi - \varphi(x_Q)|\mathrm{d}x|\Phi(0)| + 2\varepsilon\lambda(\Omega) + \int_Q |\varphi||\tilde{h} - h(x_Q)|\mathrm{d}\lambda \leqslant \delta|\Omega| + 3\varepsilon\lambda(\Omega).$$

We next focus on

$$\left|\sum_{Q\in\mathcal{F}_\delta^c}\left(|Q|\varphi(x_Q)\left(\Phi(0) + \frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty, \Phi_p^\infty\rangle\right) - \varphi(x_Q)\int_Q \Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x\right)\right| \leqslant \varepsilon|\Omega|.$$

Defining $\tilde{u}^\varepsilon := \tilde{u}_Q^\varepsilon$ on each $Q \in \mathcal{F}_\delta^c$ and extending by zero to the rest of Ω, we obtain $\tilde{u}^\varepsilon \in \mathrm{C}_c^\infty(\Omega, \mathbb{U})$. Further, we have

$$\sum_{Q\in\mathcal{F}_\delta^c} \varphi(x_Q)\int_Q \Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x = \int_{\bigcup\mathcal{F}_\delta^c} \varphi(x)\Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x + \mathcal{E}_4,$$

where

$$\begin{aligned}|\mathcal{E}_4| &\leqslant c\delta\sum_{Q\in\mathcal{F}_\delta^c}\int_Q \Phi_0(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x \leqslant c\delta\left(\varepsilon|\Omega| + \sum_{Q\in\mathcal{F}_\delta^c}|Q|\Phi_0(0) + \lambda(Q)h_0(x_Q)\right)\\ &\leqslant c\delta\left(\varepsilon|\Omega| + \int_{\bigcup\mathcal{F}_\delta^c}|h_0|\mathrm{d}\lambda\right) \leqslant c\delta\left(\varepsilon|\Omega| + \lambda(\Omega)\right)\end{aligned}$$

where the last integral can be computed explicitly. Collecting, we proved that

$$\left|\int_\Omega \varphi\mathrm{d}x\Phi(0) + \int_\Omega\langle\nu_x^\infty, \Phi_p^\infty\rangle\mathrm{d}\lambda(x) - \int_\Omega \varphi(x)\Phi(\mathcal{B}\tilde{u}_Q^\varepsilon(x))\mathrm{d}x\right| \to 0 \quad \text{as } \varepsilon\downarrow 0.$$

The proof is complete. □

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